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Infinitesimal operators for representations of complex Lie groups and Clebsch–Gordan coefficients for compact groups

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Abstract. Explicit expressions are obtained for the infinitesimal operators of the degenerate representations of the groups SL(n, C), SO(n, C) and Sp(n, C) in a discrete basis. They are used to obtain the infinitesimal operators of unitary representations of the group K $\otimes$ K in a K basis, where K is one of the groups SU(n), SO(n), Sp(n). The subgroup K is diagonally embedded into K $\otimes$ K. Matrix elements (generalised Wigner d functions) of the degenerate representations of GL(n, C) and U(n) $\otimes$ U(n) are evaluated. Clebsch–Gordan series are derived for the tensor product of irreducible representations of K which are given by one non-zero integer. The infinitesimal operators are applied to obtain recurrence relations for the Clebsch–Gordan coefficients of this tensor product. It is remarkable that they connect Clebsch–Gordan coefficients corresponding to different resulting representations.

1. Introduction

Representations of Lie groups have found wide applications in different branches of physics (elementary particle theory, atomic physics, nuclear physics, and quantum chemistry). Clebsch–Gordan coefficients, infinitesimal operators, and matrix elements of the representations are of great importance for physical applications. We need different orthonormal bases for different physical problems.

In this article we derive explicit expressions for the infinitesimal operators of the degenerate representations of the groups SL(n, C), SO(n, C) and Sp(n, C) in a K basis, where K is a maximal compact subgroup. It is clear that K = SU(n) for SL(n, C), K = SO(n) for SO(n, C) and K = Sp(n) for Sp(n, C). The representations under consideration are characterised by two numbers. The infinitesimal operators of these representations are used to obtain the infinitesimal operators of the finite-dimensional irreducible representations of the groups K $\otimes$ K. The latter representations have the highest weights $(M_1, 0, \ldots, 0)$ $(0, \ldots, 0, M_2)$ for SU(n) $\otimes$ SU(n) and $(M_1, 0, \ldots, 0)$ $(M_2, 0, \ldots, 0)$ for SO(n) $\otimes$ SO(n) and Sp(n) $\otimes$ Sp(n).

Expressions for the infinitesimal operators derived here are valid for every K basis. The formulae contain the Clebsch–Gordan coefficients of the tensor product of simple representations of K. The Clebsch–Gordan coefficients can be taken for any K basis. The infinitesimal operators correspond to the same basis.

We derive matrix elements (generalised Wigner d functions) of the degenerate representations of the groups GL(n, C) and U(n) $\otimes$ U(n) in the U(n) basis. The subgroup U(n) is diagonally embedded into U(n) ⊗ U(n). We have obtained four different expressions for the matrix elements. They are expressed with the help of the hypergeometric functions ${}_2F_1$.

The infinitesimal operators of the representations of K ⊗ K in a K basis are related to the Clebsch–Gordan coefficients for K. It leads to the recurrence formulae for the Clebsch–Gordan coefficients (Klimyk 1980). These formulae connect the Clebsch–Gordan coefficients corresponding to different resulting representations of K. Using our infinitesimal operators we obtain recurrence relations for Clebsch–Gordan coefficients for the tensor product $(M_1, 0, \ldots, 0) \otimes (0, \ldots, 0, M_2)$ of the representations of SU(n) and for the tensor product $(M_1, 0, \ldots, 0) \otimes (M_2, \ldots, 0)$ of the representations of SO(n) and Sp(n). These relations are valid for Clebsch–Gordan coefficients taken for any K basis. We have obtained Clebsch–Gordan series for these tensor products.

2. Degenerate representations of SL(n, C), SO(n, C) and Sp(n, C)

We describe some subgroups of the groups SL(n, C), SO(n, C) and Sp(n, C) which will be used to derive the degenerate representations. The Lie algebras of SL(n, C), SO(n, C) and Sp(n, C) will be denoted by sl(n, C), so(n, C) and sp(n, C) respectively. We shall use the realisations of these algebras given by Jacobson (1961). The representations of these groups are described by Gel'fand and Naimark (1950). We shall use the realisations of the degenerate representations given by Knapp and Stein (1980).

Let G denote one of the groups SL(n, C), SO(n, C), Sp(n, C). Let G = ANK be an Iwasawa decomposition of G (Barut and Raczka 1977, Warner 1972), where K is a maximal compact subgroup of G, and A is a commutative subgroup. An important subgroup of G is

$$P = ANM_1(K) = A_1N_1M_1 \tag{1}$$

where A and N are taken from the Iwasawa decomposition of G, A_1 is a one-dimensional subgroup of A, $N_1 \subset N$. The subgroup M_1 is a maximal connected subgroup of G such that $m_1a_1 = a_1m_1$, $m_1 \in M_1$, $a_1 \in A_1$, and $M_1(K) = M_1 \cap K$. Let us describe these subgroups for SL(n, C), SO(n, C) and Sp(n, C).

For SL(n, C), A consists of the diagonal matrices

$$\mathrm{diag}(t_1, t_2, \ldots, t_n) \in SL(n,C) \qquad t_i > 0. \tag{2}$$

For A_1 we have $A_1 = \exp a_1$, where a_1 is a Lie algebra of A_1. The algebra a_1 consists of the matrices

$$\mathrm{diag}\left(\frac{-t}{n-1}, \frac{-t}{n-1}, \ldots, \frac{-t}{n-1}, t\right) \qquad t \in R. \tag{3}$$

For SL(n, C)

$$M_1 = \mathrm{diag}(GL(n-1, C), GL(1, C)) \qquad M_1(K) = \mathrm{diag}(U(n-1), U(1)) \tag{4}$$

where the matrices have a unit determinant.

For SO(n, C), A consists of the matrices

$$\mathrm{diag}(t_1, t_2, \ldots, t_k, 1, t_1^{-1}, t_2^{-1}, \ldots, t_k^{-1}) \qquad t_i > 0 \tag{5}$$

if $n=2k+1$. If $n=2k$, we have to omit 1. For the subgroup $A_1=\exp a_1$ the algebra a_1 consists of the matrices

$$h_t=\mathrm{diag}(0,\ldots,0,t,0,\ldots,0,-t) \qquad t\in R. \tag{6}$$

The subgroups M_1 and $M_1(K)$ of $SO(n,C)$ are isomorphic to the groups

$$M_1\sim\mathrm{diag}(SO(n-2,C),SO(2,C)) \qquad M_1(K)\sim\mathrm{diag}(SO(n-2),SO(2)). \tag{7}$$

In order to verify it we have to use the realisations of these groups given by Jacobson (1961).

The subgroups A and A_1 of $Sp(n,C)$ are the same as for $SO(n,C)$, $n=2k$. For M_1 and $M_1(K)$ we have

$$M_1=\mathrm{diag}(Sp(n-2,C),t,t^{-1}) \qquad t\in C \tag{8}$$

$$M_1(K)=\mathrm{diag}(Sp(n-2),u,u^{-1}) \qquad u\in U(1). \tag{9}$$

The subgroup A of G can be represented as $A=A_1A_2$, where A_2 is a subgroup of A. For $SL(n,C)$ A_2 consists of the matrices (2) with $t_n=1$. For $SO(n,C)$ and $Sp(n,C)$ A_2 consists of the matrices (5) (without 1 if $n=2k$), for which $t_k=1$.

The subgroup $M_1(K)$ of G is isomorphic to a direct product of two subgroups M_2M_3, where $M_3=U(1)$ for $SL(n,C)$ and $Sp(n,C)$, and $M_3=SO(2)$ for $SO(n,C)$. Since $SO(2)\sim U(1)$, then M_3 is the same for all groups G.

Let us consider the one-dimensional representation

$$h_1h_2nm_2m_3\to\exp[\lambda(\log h_1)]\omega(m_3) \tag{10}$$

$$h_1\in A_1 \qquad h_2\in A_2 \qquad n\in N \qquad m_2\in M_2 \qquad m_3\in M_3$$

of the subgroup (1) of G, where λ is a complex linear form on the Lie algebra a_1 of the group A_1, and ω is a one-dimensional representation of $M_3\sim U(1)$. It is clear that λ is characterised by a complex number and ω by an integer.

The representation (10) of P induces the representation of G. We denote it by $\pi_{\lambda\omega}$. It can be realised in the Hilbert space $L^2_\omega(K)$, which consists of all functions of $L^2(K)$ satisfying the condition

$$f(mk)=\omega(m_3)f(k) \qquad m=m_2m_3\in M_2M_3. \tag{11}$$

The operators $\pi_{\lambda\omega}(g)$, $g\in G$, act upon $L^2_\omega(K)$ as

$$\pi_{\lambda\omega}(g)f(k)=\exp[\lambda(\log h_1)]f(k_g)$$

where $h_1\in A_1$ and $k_g\in K$ are defined by the Iwasawa decomposition of kg: $kg=h_1h_2nk_g$, $h_2\in A_2$, $n\in N$. The representations $\pi_{\lambda\omega}$ constitute the degenerate series.

3. Preliminaries

Let $\mathcal{G}$ be a Lie algebra of G. Let $\mathcal{G}=\mathcal{K}+\mathcal{P}$ be a Cartan decomposition of $\mathcal{G}$, where $\mathcal{K}$ is a Lie algebra of K (Helgason 1962). The subalgebra a_1 is contained in $\mathcal{P}$. Let $\mathcal{P}_c$ be a complexification of $\mathcal{P}$. Let us consider the pair $(\mathcal{G},a_1)$. The system of restricted roots is defined for it (Warner 1972). Since a_1 is one dimensional, there is one simple restricted root α. Let $B(\cdot,\cdot)$ be a Cartan–Killing form on $\mathcal{G}$ and θ a Cartan involution (Helgason 1962). Then

$$\langle x, y\rangle = -cB(x, \theta y) \qquad x, y \in \mathscr{G} \qquad c>0 \tag{12}$$

is a scalar product on $\mathscr{G}$ (Helgason 1962). The adjoint representation of G in $\mathscr{G}$ (i.e. the representation $g \to g^{-1}xg$, $x \in \mathscr{G}$) will be denoted by Ad.

The infinitesimal operators of the representations $\pi_{\lambda\omega}$ will be investigated by means of the following lemma (cf Klimyk 1979, lemma 5.2).

Lemma. The infinitesimal operators $\pi_{\lambda\omega}(Y)$, $Y \in \mathscr{P}_c$, act upon the infinitely differentiable functions of $L^2_\omega(\mathrm{K})$ as

$$\pi_{\lambda\omega}(Y)f(k) = \langle(\mathrm{Ad}\, k)Y, H\rangle\lambda(H)f(k) - \langle(\mathrm{Ad}\, k)Y, \rho\rangle f(k) + \tfrac{1}{2}[Q, \langle(\mathrm{Ad}\, k)Y, h\rangle]f(k) \tag{13}$$

where H is an element of a_1 for which $\langle H, H\rangle = 1$, h is an element of a_1 such that $\alpha(h) = 1$, Q is identical to the operator Q_1 of formula (5) of Klimyk and Gruber (1979), ρ is half of the sum of the positive restricted roots of the pair $(\mathscr{G}, \mathrm{a}_1)$ represented as an element of a_1 and $[\cdot, \cdot]$ denotes a commutator of Q with the multiplication operator.

We need an orthonormal basis of $L^2_\omega(\mathrm{K})$. The functions of $L^2_\omega(\mathrm{K})$ satisfy the condition (11). Therefore, the matrix elements of the irreducible representations of K, for which the relation (11) is valid, can be taken as a basis of $L^2_\omega(\mathrm{K})$. The relation (11) implies the left invariance with respect to the subgroup M_2. Hence, condition (11) can be satisfied by the matrix elements of the representations of $\mathrm{K} = \mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ with the highest weights $(m_1, m_2, 0, \ldots, 0)$, $m_1 \geqslant m_2 \geqslant 0$, and of the representations of $\mathrm{K} = \mathrm{SU}(n)$ with the highest weights $(m_1, 0, \ldots, 0, m_2)$, $m_1 \geqslant 0 \geqslant m_2$. These representations of K will be denoted by $[m_1 m_2] \equiv D^{m_1 m_2}$. The condition (11) means also that

$$f(m_3 k) = \omega(m_3) f(k) \qquad m_3 \in \mathrm{M}_3. \tag{14}$$

It implies restrictions for the integers m_1 and m_2. It is clear that the representation ω of U(1) is of the form $\mathrm{e}^{\mathrm{i}\phi} \to \mathrm{e}^{\mathrm{i}q\phi}$, where q is an integer.

For SU(n) condition (14) means that $m_1 + m_2 = q$. This follows from the formula for the operators E_{kk} in the representations of U(n) (Gel'fand and Zetlin 1950).

For Sp(n) condition (14) means that $m_1 + m_2 \geqslant |q|$ and $(-1)^{m_1+m_2} = (-1)^q$, i.e. $m_1 + m_2$ and q are of the same parity. Moreover, the representation $[m_1 m_2]$ of Sp(n) contains the representation $[0] \otimes \omega$ of $\mathrm{Sp}(n-2) \otimes \mathrm{U}(1)$ with the unit multiplicity. Here [0] denotes the representation of $\mathrm{Sp}(n-2)$ with the highest weight $(0, \ldots, 0)$. These statements follow from the reduction $\mathrm{Sp}(n) \supset \mathrm{Sp}(n-2) \otimes \mathrm{U}(1)$ for representations of Sp(n) (Zhelobenko 1970).

Lemma. Let $[m]$ be the representation $\mathrm{e}^{\mathrm{i}\phi} \to \mathrm{e}^{\mathrm{i}m\phi}$ of SO(2), and [0] the representation of $\mathrm{SO}(n-2)$ with the highest weight $(0, \ldots, 0)$. Multiplicities of the representations $[0] \otimes [m]$ of $\mathrm{SO}(n-2) \otimes \mathrm{SO}(2)$ in the representation $[m_1 m_2]$ of SO(n) do not exceed 1. Moreover, a set of the representations $[0] \otimes [m]$ of $\mathrm{SO}(n-2) \otimes \mathrm{SO}(2)$, which are contained in $[m_1 m_2]$, coincides with $[0] \otimes [m_1 - m_2 - i]$, $i = 0, 2, 4, \ldots, 2(m_1 - m_2)$.

This lemma is proved by means of a decomposition of the character of the representation $[m_1 m_2]$ into the irreducible characters of $\mathrm{SO}(n-2) \otimes \mathrm{SO}(2)$ and using the results obtained for the representations of SO(n) by Kachurik and Klimyk (1982).

It follows from this lemma that for SO(n) condition (14) means that $m_1+m_2 \geq |q|$ and the integers m_1+m_2 and q are of the same parity.

It is known that the multiplicity of the representation $[m_1m_2]$ of K in $\pi_{\lambda\omega}$ is equal to the multiplicity of the representation $[0]\otimes\omega$ of $M_1(K)$ in $[m_1m_2]$ (Gel'fand and Naimark 1950). Therefore, multiplicities of the representations of K in $\pi_{\lambda\omega}$ do not exceed 1. Moreover, restriction of the representation $\pi_{\lambda\omega}$ of SL(n, C) onto K = SU(n) contains the representations $[m_1m_2]$ of SU(n) for which $m_1+m_2=q$. A restriction of the representation $\pi_{\lambda\omega}$ of SO(n, C) or Sp(n, C) onto K contains the representations $[m_1m_2]$ of K for which $m_1+m_2\geq|q|$ and $(-1)^{m_1+m_2}=(-1)^q$.

Let $|{}^{m_1m_2}_{\omega}\rangle$ be a normalised vector of the carrier space of the representation $[m_1m_2]$ of K which transforms according to the representation $[0]\otimes\omega$ of $M_1(K)$. Let $|{}^{m_1m_2}_{r}\rangle$, $r=1,2,\ldots,$ dim $[m_1m_2]$, be any orthonormal basis of the space of the representation $[m_1m_2]$. The functions

$$(\dim[m_1m_2])^{1/2}\langle{}^{m_1m_2}_{\omega}|D^{m_1m_2}(k)|{}^{m_1m_2}_{r}\rangle \equiv |m_1m_2, r\rangle \qquad k\in K \tag{15}$$

for all r and for all $[m_1m_2]$, admitted by $\pi_{\lambda\omega}$, constitute an orthonormal basis of $L^2_\omega(K)$.

We shall find the infinitesimal operators $\pi_{\lambda\omega}(Y)$, $Y\in\mathcal{P}$, in the basis $|m_1m_2, r\rangle$. The derivation is similar to the one given by Klimyk and Gruber (1979) for the representations of the group U(p, q). Therefore, we omit details here.

The scalar product (12) can be given on $\mathcal{P}$ as

$$\langle x, y\rangle = b \operatorname{Tr} x\bar{y}^{\mathrm{T}} \tag{16}$$

where $b=1$ for SL(n, C), and $b=\frac{1}{2}$ for SO(n, C) and Sp(n, C). In (16) T denotes a transposition.

Let G = SL(n, C). According to (16) for the matrices H and h of (13) we have

$$h=\left(\frac{n-1}{n}\right)^{1/2} H=\frac{n-1}{n}e_{nn}-\frac{1}{n}(e_{11}+\ldots+e_{n-1,n-1}) \tag{17}$$

where e_{ij} is a matrix for which $(e_{ij})_{st}=\delta_{is}\delta_{jt}$. The simple restricted root of the pair (sl(n, C), a_1) is defined by the relation $\alpha(h)=(n-1)n^{-1}$. The formula $\alpha(h')=\langle h_\alpha, h'\rangle$, $h'\in a_1$, defines the correspondence between α and the element $h_\alpha\in a_1$. It is clear that $h_\alpha=(n-1)^{-1}nh$.

For SO(n,C) and Sp(n, C) the root α is defined by the relation $\alpha(h_t)=t$ where h_t is given by (6). Therefore, we have for these groups that $H=h=h_\alpha$, and this element is equal to the matrix (6) at $t=1$.

Now for the summands of the relation (13) we have

$$\langle(\operatorname{Ad} k)Y, H\rangle\lambda(H)=\langle(\operatorname{Ad} k)Y, h\rangle\lambda(h_\alpha) \tag{18}$$

$$\langle(\operatorname{Ad} k)Y, \rho\rangle=\tfrac{1}{2}(p+2s)\langle h_\alpha, h_\alpha\rangle\langle(\operatorname{Ad} k)Y, h\rangle \tag{19}$$

where p is the multiplicity of the root α, and s the multiplicity of the root 2α. For SL(n, C) $p=2(n-1)$, $s=0$. For SO(n, C) $p=2(n-2)$, $s=0$. For Sp(n, C) $p=2(n-2)$, $s=2$.

Now we consider the operator Q of (13). It is constructed by means of the subgroup chains (2) and (3) of Klimyk and Gruber (1979). Since we consider the degenerate representations of G then the chain (2) consists of two subgroups $K=K_1\supset K_2=M_1(K)$. For SL(n, C) and SO(n, C) chain (3) of Klimyk and Gruber (1979) coincides with chain (2). For Sp(n, C) the chain (3) of Klimyk and Gruber (1979) reduces to

$$\mathrm{Sp}(n) = \mathrm{K} \supset \mathrm{Sp}(n-2) \otimes \mathrm{Sp}(2) \supset \mathrm{M}_1(\mathrm{K}) = \mathrm{Sp}(n-2) \otimes \mathrm{U}(1). \tag{20}$$

The operator Q acts upon the states (15) as

$$Q|m_1m_2, r\rangle = q(m_1m_2)|m_1m_2, r\rangle \tag{21}$$

where $q(m_1m_2)$ is a number.

From (18), (19), (21) and (13) we obtain

$$\pi_{\lambda\omega}(Y)|m_1m_2, r\rangle = [\lambda(h_\alpha) - \tfrac{1}{2}(p+2s)\langle h_\alpha, h_\alpha\rangle + \tfrac{1}{2}(Q - q(m_1m_2))]\langle(\mathrm{Ad}\,k)Y, h\rangle|m_1m_2, r\rangle. \tag{22}$$

This relation will be considered for a basis of $\mathscr{P}$ consisting of orthonormal elements $Y = Y_1, Y_2, \ldots, Y_q$, $q = \dim \mathscr{P}$, with respect to the scalar product (12). Since $[\mathscr{K}, \mathscr{P}] \subset \mathscr{P}$, then $\mathscr{P}$ is a carrier space of the representation Ad of K and $\mathscr{K}$. This representation has the highest weight $(1, 0, \ldots, 0, -1)$ for SU(n), $(1, 1, 0, \ldots, 0)$ for SO(n), and $(2, 0, \ldots, 0)$ for Sp(n). For SU(n) and SO(n) the elements $Y_1, Y_2, \ldots, Y_q$ can be taken to correspond to the Gel'fand–Zetlin patterns for the representation Ad. For Sp(n) they can correspond to the patterns of Zhelobenko (1970), which are similar to the Gel'fand–Zetlin patterns.

The functions $\langle(\mathrm{Ad}\,k)Y_j, h\rangle$ are matrix elements of the representation Ad of K. Since $|m_1m_2, r\rangle$ are also matrix elements of the representations of K (cf formula (15)), then

$$\langle(\mathrm{Ad}\,k)Y_j, h\rangle\,|m_1m_2, r\rangle = \sum_{m'_1m'_2} \sum_{\gamma, r'} \left(\frac{\dim[m_1m_2]}{\dim[m'_1m'_2]}\right)^{1/2} \left\langle \begin{matrix} m_1m_2 \\ \omega \end{matrix}; \begin{matrix} \mathrm{Ad} \\ h \end{matrix} \middle| \begin{matrix} m'_1m'_2 \\ \omega \end{matrix} \right\rangle^{\gamma} \times \left\langle \begin{matrix} m'_1m'_2 \\ r' \end{matrix} \middle| \begin{matrix} m_1m_2 \\ r \end{matrix}; \begin{matrix} \mathrm{Ad} \\ Y_j \end{matrix} \right\rangle^{\gamma} |m'_1m'_2, r'\rangle \tag{23}$$

where $\langle \ldots | \ldots \rangle$ are the Clebsch–Gordan coefficients for K, and γ separates multiple representations in the tensor product $[m_1m_2] \otimes \mathrm{Ad}$. Let us note that the element h of (23) has the unit norm for SO(n, C) and Sp(n, C) and the norm $[(n-1)/n]^{1/2}$ for SL(n, C).

The linear form λ on a_1 is characterised by the complex number $\sigma = \lambda(h_\alpha)$. Let q be an integer which defines the representation ω of M_3: $e^{i\phi} \to e^{iq\phi}$. Then the representation $\pi_{\lambda\omega}$ will be denoted by $\pi^{\sigma q}$.

4. Infinitesimal operators of $\pi^{\sigma q}$ for SL(n, C)

The summation in (23) is over all vectors $|m'_1m'_2, r'\rangle$ for which the Clebsch–Gordan coefficients are not equal to zero. Let us consider the tensor product of the representations of SU(n) with the highest weights $(m_1, 0, \ldots, 0, m_2)$ and $(1, 0, \ldots, 0, -1)$. It contains the representations of SU(n) with the highest weights $(m_1, 0, \ldots, 0, m_2)$ (with the multiplicity 2), $(m_1+1, 0, \ldots, 0, m_2-1)$, $(m_1-1, 0, \ldots, 0, m_2+1)$. Other highest weights contain three or four non-zero coordinates, or do not satisfy the condition $m'_1 + m'_2 = q$; we are not interested in them. Therefore, the summation in (23) is over

$$(m'_1, m'_2) = (m_1, m_2), (m_1+1, m_2-1), (m_1-1, m_2+1). \tag{24}$$

We have to substitute (23) into (22) and to find eigenvalues of the operator $Q-q(m_1m_2)$. These eigenvalues are evaluated in the same way as in the case of the group U(p, q) (Klimyk and Gruber 1979). We have

$$\tfrac{1}{2}(Q-q(m_1m_2))|m_1+1, m_2-1, r\rangle=(m_1-m_2+n)|m_1+1, m_2-1, r\rangle$$

$$\tfrac{1}{2}(Q-q(m_1m_2))|m_1-1, m_2+1, r\rangle=(m_2-m_1-n+2)|m_1-1, m_2+1, r\rangle$$

$$\tfrac{1}{2}(Q-q(m_1m_2))|m_1, m_2, r\rangle=0.$$

Therefore, for SL(n, C), we obtain that

$$\begin{aligned}\pi^{\sigma q}(Y_j)|m_1, m_2, r\rangle &= \sum_{r'}(\sigma+m_1-m_2)K(m_1+1, m_2-1, r', j)|m_1+1, m_2-1, r'\rangle\\ &\quad+\sum_{r'}(\sigma-m_1+m_2-2n+2)K(m_1-1, m_2+1, r', j)|m_1-1, m_2+1, r'\rangle\\ &\quad+\sum_{r'}(\sigma-n)K(m_1, m_2, r', j)|m_1, m_2, r'\rangle \end{aligned}\tag{25}$$

where

$$\begin{aligned}K(m_1', m_2', r', j) &= \left(\frac{\dim[m_1m_2]}{\dim[m_1'm_2']}\right)^{1/2}\sum_\gamma\left\langle\begin{matrix}m_1m_2\\ \omega\end{matrix};\begin{matrix}1, -1\\ h\end{matrix}\middle|\begin{matrix}m_1'm_2'\\ \omega\end{matrix}\right\rangle^\gamma\\ &\quad\times\left\langle\begin{matrix}m_1'm_2'\\ r'\end{matrix}\middle|\begin{matrix}m_1m_2\\ r\end{matrix};\begin{matrix}1, -1\\ Y_j\end{matrix}\right\rangle^\gamma.\end{aligned}\tag{26}$$

The summation in (25) is over all r' which are admitted by the Clebsch–Gordan coefficients. In (26) there are two summands if $(m_1', m_2')=(m_1, m_2)$, and one summand if $(m_1', m_2')\neq(m_1, m_2)$.

Thus, we have the infinitesimal operators $\pi^{\sigma q}(Y_j)$ of the representations $\pi^{\sigma q}$ of SL(n, C) in SU(n) bases. They allow us to separate irreducible representations in the set of all representations $\pi^{\sigma q}$ and to obtain a structure of reducible representations $\pi^{\sigma q}$. It can be done exactly in the same way as for the groups U(n, 1) and SO_0 (n, 1) (Klimyk 1979). Therefore, we formulate the theorem without proof.

Theorem. The representation $\pi^{\sigma q}$ is irreducible if and only if σ is not an integer such that $(-1)^q=(-1)^\sigma$, or $-|q|<\sigma<2n+|q|$. If $\sigma\leqslant-|q|$ and σ is an integer such that $(-1)^q=(-1)^\sigma$, then $\pi^{\sigma q}$ contains two (and only two) irreducible representations of SL(n,C). One of them is finite dimensional. It will be denoted by $F^{\sigma q}$. A restriction of $F^{\sigma q}$ onto SU(n) contains (with the unit multiplicity) all representations $[m_1m_2]$ of SU(n) for which $m_1+m_2=q$ and $m_1-m_2\leqslant-\sigma$, and only them.

Let us note that the vanishing of the multiplier $(\sigma+m_1-m_2)$ of the first summand of the right-hand side of (25) leads to a separation of the finite-dimensional subrepresentation $F^{\sigma q}$ in $\pi^{\sigma q}$. It is clear that the infinitesimal operators of $F^{\sigma q}$ are given by (25).

It can be seen that $F^{\sigma q}$ is a real analytic representation of SL(n, C). Therefore, this representation is the tensor product of complex analytic and complex anti-analytic representations of SL(n, C) (Zhelobenko 1970, ch 6). Thus, the representation $F^{\sigma q}$ is given by the two highest weights $((-\sigma+q)/2, 0, \ldots, 0)((-\sigma-q)/2, 0, \ldots, 0)$. An explicit expression for these highest weights is defined by theorem 5.13a of Klimyk (1979).

The representations $\pi^{\sigma q}$ are unitary if $\sigma-n$ is imaginary. These representations constitute the principal degenerate series of SL(n,C).

5. Infinitesimal operators of $\pi^{\sigma q}$ for SO(n, C)

In this case the considerations are the same as for SL(n, C). Decomposing the tensor product of the representations of SO(n) with the highest weights $(m_1, m_2, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$ into irreducible representations, we find that the summation in (23) is over

$$(m_1', m_2') = (m_1+1, m_2+1), (m_1+1, m_2-1),$$
$$(m_1-1, m_2+1), (m_1-1, m_2-1), (m_1, m_2). \tag{27}$$

This tensor product contains the representation $[m_1 m_2]$ with multiplicity 2 and other representations with multiplicity 1. Evaluating eigenvalues of the operator $Q - q(m_1 m_2)$, we find from (22) and (23) that

$$\begin{aligned}\pi^{\sigma q}(Y_j)|m_1, m_2, r\rangle &= \sum_{r'} (\sigma + m_1 + m_2) K(m_1+1, m_2+1, r', j)|m_1+1, m_2+1, r'\rangle \\ &+ \sum_{r'} (\sigma + m_1 - m_2 - n + 4) K(m_1+1, m_2-1, r', j)|m_1+1, m_2-1, r'\rangle \\ &+ \sum_{r'} (\sigma - m_1 + m_2 - n + 2) K(m_1-1, m_2+1, r', j)|m_1-1, m_2+1, r'\rangle \\ &+ \sum_{r'} (\sigma - m_1 - m_2 - 2n + 6) K(m_1-1, m_2-1, r', j)|m_1-1, m_2-1, r'\rangle \\ &+ \sum_{r'} (\sigma - n + 2) K(m_1, m_2, r', j)|m_1, m_2, r'\rangle \end{aligned} \tag{28}$$

where

$$K(m_1', m_2', r', j) = \sum_{\gamma} \left(\frac{\dim[m_1 m_2]}{\dim[m_1' m_2']}\right)^{1/2} \times \left\langle \begin{matrix} m_1 m_2 \\ \omega \end{matrix}; \begin{matrix} 1, 1 \\ h \end{matrix} \middle| \begin{matrix} m_1' m_2' \\ \omega \end{matrix} \right\rangle^{\gamma} \left\langle \begin{matrix} m_1' m_2' \\ r' \end{matrix} \middle| \begin{matrix} m_1 m_2 \\ r \end{matrix}; \begin{matrix} 1, 1 \\ Y_j \end{matrix} \right\rangle^{\gamma}. \tag{29}$$

In (29) there are two summands if $(m_1' m_2') = (m_1 m_2)$, and one summand if $(m_1' m_2') \neq (m_1 m_2)$.

Theorem. The representation $\pi^{\sigma q}$ of SO(n, C) is irreducible if and only if σ is not an integer such that $(-1)^{\sigma} = (-1)^q$, or $-|q| < \sigma < 2n - 4 + |q|$. If $\sigma \leq -|q|$ and $(-1)^{\sigma} = (-1)^q$, then $\pi^{\sigma q}$ contains two (and only two) irreducible representations of SO(n, C). One of them is finite dimensional. It will be denoted by $F^{\sigma q}$. A restriction of $F^{\sigma q}$ onto SO(n) contains (with the unit multiplicity) all representations $[m_1 m_2]$ of SO(n) for which $m_1 - m_2 \geq |q|$ and $m_1 + m_2 \leq -\sigma$ and only them.

The vanishing of the multiplier $(\sigma + m_1 + m_2)$ of the first summand of the right-hand side of (28) is a reason for the separation of the finite-dimensional subrepresentation $F^{\sigma q}$ in $\pi^{\sigma q}$. It is clear that the infinitesimal operators of $F^{\sigma q}$ are given by (28). The representation $F^{\sigma q}$ is given by the two highest weights $((-\sigma+q)/2, 0, \ldots, 0)$ $((-\sigma-q)/2, 0, \ldots, 0)$.

The representations $\pi^{\sigma q}$, for which $\sigma - n + 2$ are imaginary, constitute the principal degenerate unitary series of SO(n, C).

6. Infinitesimal operators of $\pi^{\sigma q}$ for Sp(n, C)

Decomposing the tensor product of the representations of Sp(n) with the highest weights $(m_1, m_2, 0, \ldots, 0)$ and $(2, 0, \ldots, 0)$ into irreducible representations, we find that for Sp(n, C) the summation in (23) is over

$$(m_1', m_2') = (m_1+2, m_2), (m_1, m_2+2), (m_1-2, m_2), (m_1, m_2-2), (m_1+1, m_2+1),$$
$$(m_1+1, m_2-1), (m_1-1, m_2+1), (m_1-1, m_2-1), (m_1, m_2).$$

The decomposition contains $[m_1 m_2]$ with multiplicity 2 and other representations with multiplicity 1. Evaluating eigenvalues of the operator $Q - q(m_1 m_2)$, we find from (22) and (23) that

$$\begin{aligned}
\pi^{\sigma q}(Y_j)|m_1, m_2, r\rangle
&= \sum_{r'} (\sigma + m_1 + m_2) K(m_1+2, m_2, r', j)|m_1+2, m_2, r'\rangle \\
&+ \sum_{r'} (\sigma + m_1 + m_2) K(m_1, m_2+2, r', j)|m_1, m_2+2, r'\rangle \\
&+ \sum_{r'} (\sigma + m_1 + m_2) K(m_1+1, m_2+1, r', j)|m_1+1, m_2+1, r'\rangle \\
&+ \sum_{r'} (\sigma - m_1 - m_2 - 2n + 2) K(m_1-2, m_2, r', j)|m_1-2, m_2, r'\rangle \\
&+ \sum_{r'} (\sigma - m_1 - m_2 - 2n + 2) K(m_1, m_2-2, r', j)|m_1, m_2-2, r'\rangle \\
&+ \sum_{r'} (\sigma - m_1 - m_2 - 2n + 2) K(m_1-1, m_2-1, r', j)|m_1-1, m_2-1, r'\rangle \\
&+ \sum_{r'} (\sigma - n) K(m_1+1, m_2-1, r', j)|m_1+1, m_2-1, r'\rangle \\
&+ \sum_{r'} (\sigma - n) K(m_1-1, m_2+1, r', j)|m_1-1, m_2+1, r'\rangle \\
&+ \sum_{r'} (\sigma - n) K(m_1, m_2, r', j)|m_1, m_2, r'\rangle,
\end{aligned} \tag{30}$$

where

$$K(m_1', m_2', r', j) = \left(\frac{\dim [m_1 m_2]}{\dim [m_1' m_2']}\right)^{1/2} \times \sum_\gamma \left\langle \begin{matrix} m_1 m_2 \\ \omega \end{matrix}; \begin{matrix} 2, 0 \\ h \end{matrix} \middle| \begin{matrix} m_1' m_2' \\ \omega \end{matrix} \right\rangle^\gamma \left\langle \begin{matrix} m_1' m_2' \\ r' \end{matrix} \middle| \begin{matrix} m_1 m_2 \\ r \end{matrix}; \begin{matrix} 2, 0 \\ Y_j \end{matrix} \right\rangle^\gamma. \tag{31}$$

The summation in (30) is over all r' which are admitted by Clebsch–Gordan coefficients. In (31) there are two summands if $(m_1', m_2') = (m_1, m_2)$, and one summand if $(m_1', m_2') \neq (m_1, m_2)$.

Theorem. The representation $\pi^{\sigma q}$ of Sp(n, C) is irreducible if and only if σ is not an integer such that $(-1)^\sigma = (-1)^q$, or $-|q| < \sigma < 2n + |q|$. If $\sigma \leq -|q|$ and $(-1)^\sigma = (-1)^q$, then $\pi^{\sigma q}$ contains two (and only two) irreducible representations of Sp(n, C). One of them is finite dimensional (we denote it by $F^{\sigma q}$). A restriction of $F^{\sigma q}$ onto Sp(n) contains (with unit multiplicity) all representations $[m_1 m_2]$ of Sp(n) for which $m_1 - m_2 \geq |q|$ and $m_1 + m_2 \leq -\sigma$, and only them.

The representation $F^{\sigma q}$ is given by the two highest weights $((-\sigma+q)/2, 0, \ldots, 0)$ $((-\sigma-q)/2, 0, \ldots, 0)$. Its infinitesimal operators are given by the formula (30).

The representations $\pi^{\sigma q}$, for which $\sigma-n$ are imaginary, constitute the principal degenerate unitary series of $\mathrm{Sp}(n, C)$.

7. Infinitesimal operators of unitary representations of K ⊗ K in a K basis (diagonal embedding)

As above, let G denote one of the groups $\mathrm{SL}(n, C)$, $\mathrm{SO}(n, C)$, $\mathrm{Sp}(n, C)$, and let K be its maximal compact subgroup. We consider the finite-dimensional subrepresentations $F^{\sigma q}$ of the representations $\pi^{\sigma q}$ of G.

Let $\mathscr{G}=\mathscr{K}+\mathscr{P}$ be a Cartan decomposition of the Lie algebra $\mathscr{G}$ of G. The compact Lie algebra $\mathscr{K}\oplus\mathscr{K}$ corresponds to the algebra $\mathscr{G}$ (Helgason 1962). It has the decomposition

$$\mathscr{K}\oplus\mathscr{K}=\mathscr{K}+\mathrm{i}\mathscr{P} \qquad \mathrm{i}=\sqrt{-1}. \tag{32}$$

It is clear that $\mathscr{K}\oplus\mathscr{K}$ is a Lie algebra of the group K ⊗ K. By means of an analytic continuation of appropriate parameters, we obtain the finite-dimensional representation $F^{\sigma q}$ of K ⊗ K from the representation $F^{\sigma q}$ of G. Multiplying the infinitesimal operators $Y_j \in \mathscr{P}$ of the representations $F^{\sigma q}$ of G by i, we obtain the infinitesimal operators $J_j=\mathrm{i}Y_j$ of the representations $F^{\sigma q}$ of K ⊗ K. In the basis $|m_1, m_2, r\rangle$ they are given by formulae (25), (28) and (30). Their matrices do not satisfy the unitarity condition $J_j^*=-J_j$. To satisfy this condition we have to use the new basis

$$|m_1, m_2, r\rangle'=[a(m_1, m_2)]^{-1/2}|m_1, m_2, r\rangle. \tag{33}$$

For $\mathrm{SL}(n, C)$ the numbers $a(m_1, m_2)$ are defined by

$$a(m_1+i, m_2-i)=\prod_{j=0}^{i-1}\frac{-\sigma+2n+m_1-m_2+2j}{\sigma+m_1-m_2+2j}a(m_1, m_2). \tag{34}$$

For $\mathrm{SO}(n, C)$

$$a(m_1+i, m_2+i)=\prod_{j=1}^{i}\frac{-\sigma+2n+m_1+m_2+2j-6}{\sigma+m_1+m_2+2j-2}a(m_1, m_2) \tag{35}$$

$$a(m_1+i, m_2-i)=\prod_{j=1}^{i}\frac{-\sigma+n+m_1-m_2+2j-2}{\sigma-n+m_1-m_2+2j+2}a(m_1, m_2). \tag{36}$$

For $\mathrm{Sp}(n, C)$ the numbers $a(m_1, m_2)$ depend only on the sum m_1+m_2: $a(m_1, m_2)=a(m_1+m_2)$. They are defined by

$$a(m_1+m_2+2i)=\prod_{j=0}^{i-1}\frac{-\sigma+2n+m_1+m_2+2j}{\sigma+m_1+m_2+2j}a(m_1+m_2). \tag{37}$$

If we give $a(m_1, m_2)$ for fixed $m_1=m_1^\circ$, $m_2=m_2^\circ$, we obtain $a(m_1, m_2)$ for all m_1, m_2 (Klimyk 1979, ch 5).

The representation $F^{\sigma q}$ of $\mathrm{SL}(n, C)$ leads to the irreducible representation of $\mathrm{SU}(n)\otimes\mathrm{SU}(n)$ with the highest weight $(\mathrm{M}_1, 0, \ldots, 0)(0, \ldots, 0, \mathrm{M}_2)$, $\mathrm{M}_1=(-\sigma+q)/2$, $\mathrm{M}_2=(\sigma+q)/2$. From (25), (33), (34) we obtain the infinitesimal operators of this representation of $SU(n)\otimes SU(n)$ in the basis (33)

$$\begin{aligned} J_j|m_1, m_2, r\rangle = & -\sum_{r'}[(M_1-M_2-m_1+m_2)(M_1-M_2+m_1-m_2+2n)]^{1/2} \\ & \times K(m_1+1, m_2-1, r', j)|m_1+1, m_2-1, r'\rangle + \sum_{r'}[(M_1-M_2-m_1+m_2+2) \\ & \times(M_1-M_2+m_1-m_2+2n-2)]^{1/2}K(m_1-1, m_2+1, r', j) \\ & \times|m_1-1, m_2+1, r'\rangle \\ & -\sum_{r'} i(M_1-M_2+n)K(m_1, m_2, r', j)|m_1, m_2, r'\rangle \end{aligned} \tag{38}$$

where $K(\ldots)$ are defined by (26).

The representation $F^{\sigma q}$ of $SO(n, C)$ leads to the irreducible representation of $SO(n)\otimes SO(n)$ with the highest weight $(M_1, 0, \ldots, 0)(M_2, 0, \ldots, 0)$, $M_1 = (-\sigma+q)/2$, $M_2 = (-\sigma-q)/2$. From (28), (33), (35) and (36) we obtain the infinitesimal operators of this representation of $SO(n)\otimes SO(n)$ in the basis (33). They can be obtained from (28) if we replace $\pi^{\sigma q}(Y_j)$ by J_j and

$(\sigma+m_1-m_2-n+4)$ by $-[(M_1+M_2-m_1+m_2+n-4)(M_1+M_2+m_1-m_2+n)]^{1/2}$

$(\sigma-m_1+m_2-n+2)$ by $[(M_1+M_2+m_1-m_2+n-2) \times(M_1+M_2-m_1+m_2+n-2)]^{1/2}$

$(\sigma-m_1-m_2-2n+6)$ by $[(M_1+M_2-m_1-m_2+2)(M_1+M_2+m_1+m_2+2n-6)]^{1/2}$

$(\sigma-n+2)$ by $-i(M_1+M_2+n-2)$.

The representation $F^{\sigma q}$ of $Sp(n, C)$ leads to the irreducible representation of $Sp(n)\otimes Sp(n)$ with the highest weight $(M_1, 0, \ldots, 0)(M_2, 0, \ldots, 0)$, $M_1 = (-\sigma+q)/2$, $M_2 = (-\sigma-q)/2$. From (30), (33) and (37) we obtain the infinitesimal operators of this representation of $Sp(n)\otimes Sp(n)$ in this basis (33). They can be obtained from (30), if we replace $\pi^{\sigma q}(Y_j)$ by J_j and

$(\sigma+m_1+m_2)$ by $-[(M_1+M_2-m_1-m_2)(M_1+M_2+m_1+m_2+2n)]^{1/2}$

$(\sigma-m_1-m_2-2n+2)$ by $[(M_1+M_2-m_1-m_2+2)(M_1+M_2+m_1+m_2+2n-2)]^{1/2}$

$(\sigma-n)$ by $-i(M_1+M_2+n)$.

It is shown by direct computation that the infinitesimal operators J_j in the basis (33) satisfy the unitarity condition $J_j^* = -J_j$.

We have obtained the infinitesimal operators of the representations $F^{\sigma q}$ of $K\otimes K$ in a K basis. Let us consider the embedding of the Lie algebra $\mathscr{K}$ of K into the Lie algebra $\mathscr{K}\oplus\mathscr{K}$. Let I_s, $s = 1, 2, \ldots, \dim \mathscr{K}$, be basis elements of the subalgebra $\mathscr{K}$ from $\mathscr{G} = \mathscr{K}+\mathscr{P}$. Then $Y_s = iI_s$, $i = \sqrt{-1}$, is a basis of $\mathscr{P}$. Independent operators correspond to the elements I_s and Y_s in the representations $\pi^{\sigma q}$ and $F^{\sigma q}$. Therefore, we consider that the elements I_s and Y_s of $\mathscr{G}$ are linearly independent. The decomposition (32) of $\mathscr{K}\oplus\mathscr{K}$ corresponds to the decomposition $\mathscr{G} = \mathscr{K}+\mathscr{P}$ of $\mathscr{G}$. In this reason for the decomposition (32) we have the elements $J_s = iY_s$ instead of Y_s. The elements J_s constitute a basis of $i\mathscr{P}$. We consider that the elements I_s and J_s of $\mathscr{K}\oplus\mathscr{K}$ are also independent. Two subalgebras $\mathscr{K}$ of the left-hand side of (32) have the following bases:

$$\{X_s = \tfrac{1}{2}(I_s+J_s)\} \qquad \{X'_s = \tfrac{1}{2}(I_s-J_s)\}. \tag{39}$$

It is easy to verify that $[X_s, X'_r]=0$ for all s and r. The diagonal element $X_s+X'_s=I_s$ of the subalgebra $\mathscr{K}$ from the right-hand side of (32) corresponds to the diagonal element $X_s\oplus X'_s$ of $\mathscr{K}\oplus\mathscr{K}$. The element $X_s-X'_s=J_s$ corresponds to the element $X_s\oplus(-X'_s)$ of $i\mathscr{P}\subset\mathscr{K}\oplus\mathscr{K}$.

8. Matrix elements of the representations of GL(n, C) and U(n) ⊗ U(n) in the U(n) basis

It will be more convenient to consider the groups GL(n, C) and U(n) ⊗ U(n) instead of SL(n, C) and SU(n) ⊗ SU(n). The decomposition GL(n, C) = SL(n, C) ⊗ C_0 is valid, where C_0 is the multiplicative group of complex numbers without 0. Let us consider the subgroup

$$P=A'NM'_1(K)=A'_1N_1M'_1$$

of GL(n, C) where N and N_1 are the same as in (1), A′ consists of the matrices $\mathrm{diag}(t_1,\ldots,t_n)$, $t_j>0$, A'_1 consists of the matrices $\mathrm{diag}(1,1,\ldots,1,t)$, $t>0$, and

$$M'_1(K)=\mathrm{diag}(U(n-1),U(1))=M'_2M'_3 \qquad M'_3=U(1).$$

For the group A′ we have the decomposition $A'=A'_1A'_2$, where A'_2 consists of the matrices $\mathrm{diag}(t_1,\ldots,t_{n-1},1)$.

Let ω be the representation $e^{i\phi}\to e^{iq\phi}$ of U(1). Let $L^2_\omega(K)$, K = U(n), be a subspace of $L^2(K)$ consisting of the functions f for which

$$f(mk)=\omega(m_3)f(k) \qquad m=m_2m_3\in M'_2M'_3. \tag{40}$$

If λ is a complex linear form on the Lie algebra a'_1 of A'_1 and $\sigma=\lambda(e)$, $e=\mathrm{diag}(0,\ldots,0,1)\in a'_1$, then we define the representation $\pi^{\sigma q}$ of GL(n, C) which acts upon $L^2_\omega(K)$ by the formula

$$\pi^{\sigma q}(g)f(k)=\exp[\lambda(\log h_1)]f(k_g).$$

Here $h_1\in A'_1$ and $k_g\in K=U(n)$ are defined by the decomposition $kg=h_1h_2nk_g$, $h_2\in A'_2$, $n\in N$. The representation $\pi^{\sigma q}$ of GL(n, C) is decomposed into the product of the representation $\pi^{\sigma q}$ of SL(n, C), defined in § 2, and a one-dimensional representation of C_0.

The quotient space U($n-1$)\U(n) can be parametrised by the angles $\phi_1,\ldots,\phi_n,\theta_2,\ldots,\theta_n$ (Klimyk and Gavrilik 1979). Due to (40) functions of $L^2_\omega(K)$ can be considered as functions of these angles.

Every element $g\in GL(n,C)$ can be represented as a product of the matrices $a_t=\mathrm{diag}(1,\ldots,1,t)$, $t>0$, and elements of U(n). The representation matrix elements for U(n) are known for the Gel'fand–Zetlin basis. Therefore, we have to find the matrix elements corresponding to the matrices a_t. We obtain that

$$\pi^{\sigma q}(a_t)f(\phi_1,\ldots,\phi_n,\theta_2,\ldots,\theta_n)=\mu^{\sigma}f(\phi_1,\ldots,\phi_n,\theta_2,\ldots,\theta_{n-1},\theta'_n)$$

where

$$\mu=t\left(1+\frac{1-t^2}{t^2}\sin^2\theta_n\right)^{1/2} \qquad \sin\theta'_n=\mu^{-1}\sin\theta_n.$$

A restriction of $\pi^{\sigma q}$ onto U(n) consists of all representations of U(n) with the highest weights $(m_1, 0, \ldots, 0, m_2)$ for which $m_1+m_2=q$. Multiplicities are equal to 1. Therefore, for the matrix elements of the operators $\pi^{\sigma q}(a_t)$ we have

$$d^{\sigma q}_{(m_1 m_2)(m'_1 m'_2)(n_1 n_2)}(t) = A\int_0^{\pi/2} d\theta(\sin\theta)^{2n-3}\cos\theta[d^{(m_10\ldots0m_2)}_{(0\ldots0)(n_10\ldots0n_2)}(\theta)]^* \times \mu^{\sigma} d^{(m'_10\ldots0m'_2)}_{(0\ldots0)(n_10\ldots0n_2)}(\theta') \qquad (41)$$

where

$$A=(n-1)(\dim[m_1m_2]/\dim[m'_1m'_2])^{1/2}$$

and * denotes a complex conjugation. Here $d^{\cdots}_{\cdots}(\theta)$ are representation matrix elements for U(n), defined by formulae (48) and (49) of Klimyk and Gavrilik (1979). They can be represented as

$$d^{(m_10\ldots0m_2)}_{(0\ldots0)(n_10\ldots0n_2)}(\theta) = (\cos\theta)^{n_1+n_2-m_1-m_2}\sum_{k=n_1}^{m_1} M^{m_1m_2k}_{n_1n_2}(\sin\theta)^{2k-n_1-n_2} \qquad (42)$$

$$= (\cos\theta)^{m_1+m_2-n_1-n_2}\sum_{k=m_2}^{n_2} M^{m_1m_2k}_{n_1n_2}(\sin\theta)^{n_1+n_2-2k}. \qquad (43)$$

The expressions for N and M are given by Klimyk and Gavrilik (1979).

Substituting (42) and (43) into (41) and using formula 3.681(1) of Gradshtein and Ryzhik (1965), we obtain four expressions for the matrix elements (41)

$$\begin{aligned} d^{\sigma q}_{(m_1 m_2)(m'_1 m'_2)(n_1 n_2)}(t) &= \tfrac{1}{2}At^{\sigma+n_1+n_2}\sum_{k=n_1}^{m_1}\sum_{k'=n_1}^{m'_1} N^{m_1m_2k}_{n_1n_2}N^{m'_1m'_2k'}_{n_1n_2}t^{-2k'} \\ &\quad\times B(r,s)\,{}_2F_1\left(\frac{\sigma+q-2k'}{2}, r; r+s; \frac{t^2-1}{t^2}\right) \\ &= \tfrac{1}{2}At^{\sigma-n_1-n_2}\sum_{k=n_1}^{m_1}\sum_{k'=m'_2}^{n_2} N^{m_1m_2k}_{n_1n_2}M^{m'_1m'_2k'}_{n_1n_2}t^{2k'} \\ &\quad\times B(r',1)\,{}_2F_1\left(\frac{\sigma-q+2k'}{2}, r'; r'+1; \frac{t^2-1}{t^2}\right) \\ &= \tfrac{1}{2}At^{\sigma+n_1+n_2}\sum_{k=m_2}^{n_2}\sum_{k'=n_1}^{m'_1} M^{m_1m_2k}_{n_1n_2}N^{m'_1m'_2k'}_{n_1n_2}t^{-2k'} \\ &\quad\times B(b,1)\,{}_2F_1\left(\frac{\sigma+q-2k'}{2}, b; b+1; \frac{t^2-1}{t^2}\right) \\ &= \tfrac{1}{2}At^{\sigma-n_1-n_2}\sum_{k=m_2}^{n_2}\sum_{k'=m'_2}^{n_2} M^{m_1m_2k}_{n_1n_2}M^{m'_1m'_2k'}_{n_1n_2}t^{2k'} \\ &\quad\times B(b',d)\,{}_2F_1\left(\frac{\sigma-q+2k'}{2}, b'; b'+d; \frac{t^2-1}{t^2}\right) \end{aligned} \qquad (44)$$

where

$$r=k+k'-n_1-n_2+n-1 \qquad s=n_1+n_2+1 \qquad r'=k-k'+n-1$$
$$b=-k+k'+n-1 \qquad b'=n_1+n_2-k-k'+n-1 \qquad d=-n_1-n_2+1.$$

An analytic continuation $t\to e^{i\phi}$ in the matrices a_t gives the matrices $a(e^{i\phi})$ of the group $U(n)\otimes U(n)$. We wish to find matrix elements of the unitary irreducible representations $D^{M_1M_2}$ of $U(n)\otimes U(n)$ with the highest weights $(M_1, 0, \ldots, 0)$ $(0, \ldots, 0, M_2)$, $M_1\geq 0\geq M_2$. These representations can be obtained from the finite-dimensional representations of $GL(n, C)$ which are subrepresentations of the representations $\pi^{\sigma q}$ of $GL(n, C)$ with $\sigma = M_2-M_1$, $q = M_1+M_2$ (cf § 4). A restriction of the representation $D^{M_1M_2}$ onto $U(n)$ (diagonal embedding into $U(n)\otimes U(n)$) contains with the unit multiplicity the representations $(m_1, 0, \ldots, 0, m_2)$ of $U(n)$ for which $m_1+m_2 = M_1+M_2$, $m_1-m_2\leq M_1-M_2$. To obtain the matrix elements of $D^{M_1M_2}$ from formulae (44), we have to do an analytic continuation $t\to e^{i\phi}$ and then replace the basis $|m_1, m_2, r\rangle$ by the basis (33). As a result we obtain for them the formula

$$D^{M_1M_2}_{(m_1m_2)(m_1'm_2')(n_1n_2)}(e^{i\phi}) = Q d^{M_2-M_1, M_1+M_2}_{(m_1m_2)(m_1'm_2')(n_1n_2)}(e^{i\phi})$$

where

$$Q=\left(\prod_{s=1}^{M_1-m_1}\frac{2m_1-2M_1-2s}{2n+2m_1-2M_2-2s}\prod_{s=1}^{M_1-m_1'}\frac{2n+2m_1'-2M_2-2s}{2m_1'-2M_1-2s}\right)^{1/2}$$

and $d^{\cdots}_{\cdots}(e^{i\phi})$ are given by (44).

9. Recurrence relations for Clebsch–Gordan coefficients

The subalgebra $\mathcal{K}$ of the right-hand side of (32) is diagonally embedded into $\mathcal{K}\oplus\mathcal{K}$. Therefore, a restriction of the irreducible finite-dimensional representation of $\mathcal{K}\oplus\mathcal{K}$ with the highest weight $(m_1, m_2, \ldots, m_k)(m_1', m_2', \ldots, m_k')$ onto this diagonal subalgebra $\mathcal{K}$ leads to the tensor product of the irreducible representations of $\mathcal{K}$ with the highest weights $(m_1, m_2, \ldots, m_k)$ and $(m_1', m_2', \ldots, m_k')$. We know all irreducible representations of $\mathcal{K}$ which are contained in the representations $F^{\sigma q}$ of $\mathcal{K}\oplus\mathcal{K}$ (cf §§ 4, 5, 6). Therefore, we know the Clebsch–Gordan series for the tensor product of the representations of $SU(n)$ with the highest weights $(M_1, 0, \ldots, 0)$, $(0, \ldots, 0, M_2)$, $M_1\geq 0\geq M_2$

$$(M_1, 0, \ldots, 0)\otimes(0, \ldots, 0, M_2) = \sum_{i=0}^{\min(M_1, -M_2)}(M_1-i, 0, \ldots, 0, M_2+i) \tag{45}$$

and for the tensor product of the representations of $SO(n)$ and $Sp(n)$ with the highest weights $(M_1, 0, \ldots, 0)$ and $(M_2, 0, \ldots, 0)$, $M_1\geq M_2\geq 0$

$$(M_1, 0, \ldots, 0)\otimes(M_2, 0, \ldots, 0) = \sum_{i=0}^{M_2}\sum_{j=0}^{M_2-i}(M_1+M_2-i-2j, i, 0, \ldots, 0). \tag{46}$$

Multiplicities of representations in the decompositions (45) and (46) are equal to unity.

Now we can derive recurrence relations for Clebsch–Gordan coefficients of the decompositions (45) and (46). For this purpose we use the formula (8) of Klimyk (1980). As above, K is one of the groups $SU(n)$, $SO(n)$, $Sp(n)$. Let $\left|{M_1 \atop \alpha_1}\right\rangle$ be an orthonormal basis of the carrier space for the representation of K with the highest weight $(M_1, 0, \ldots, 0)$, and $\left|{M_2 \atop \alpha_2}\right\rangle$ for the representation of $SU(n)$ with the highest weight

$(0,\ldots,0,M_2)$ or of SO(n) and Sp(n) with the highest weight $(M_2,0,\ldots,0)$. Let $|{}^{m_1m_2}_{r}\rangle$ be the basis elements which in § 7 were denoted by $|m_1,m_2,r\rangle$ (cf formula (33)). The choice of these orthonormal bases is not restricted by any conditions. The following recurrence relations for the Clebsch–Gordan coefficients of the decompositions (45) and (46) follow from formula (8) of Klimyk (1980)

$$\sum_{m_1'm_2'r'} \left\langle \begin{matrix} M_1 & M_2 \\ \alpha_1 & \alpha_2 \end{matrix} \middle| \begin{matrix} m_1'm_2' \\ r' \end{matrix} \right\rangle \left\langle \begin{matrix} m_1'm_2' \\ r' \end{matrix} \middle| J_j \middle| \begin{matrix} m_1m_2 \\ r \end{matrix} \right\rangle$$

$$= -\sum_{\alpha_1'\alpha_2'} \left\langle \begin{matrix} M_1 \\ \alpha_1 \end{matrix} \middle| X_j \middle| \begin{matrix} M_1 \\ \alpha_1' \end{matrix} \right\rangle \left\langle \begin{matrix} M_2 \\ \alpha_2 \end{matrix} \middle| X_j' \middle| \begin{matrix} M_2 \\ \alpha_2' \end{matrix} \right\rangle \left\langle \begin{matrix} M_1 & M_2 \\ \alpha_1' & \alpha_2' \end{matrix} \middle| \begin{matrix} m_1m_2 \\ r \end{matrix} \right\rangle \tag{47}$$

where X_j and X_j' are the basis elements from (39). The left-hand side of (47) contains the matrix elements of the infinitesimal operators J_j obtained in § 7. The right-hand side of (47) contains the matrix elements of the infinitesimal operators of the irreducible representations of K. The summation on the left-hand side is the same as in the formulae for the infinitesimal operators J_j in § 7.

Formulae (47) define all the Clebsch–Gordan coefficients of the decompositions (45) and (46) up to one constant (Klimyk 1980). They relate the Clebsch–Gordan coefficients corresponding to different resulting representations.

10. Conclusions

We have obtained infinitesimal operators of the representations $\pi^{\sigma q}$ and $F^{\sigma q}$ of the complex simple Lie groups G and of the irreducible representations $F^{\sigma q}$ of the compact Lie groups $K\otimes K$ in a K basis. The expressions for infinitesimal operators include simple Clebsch–Gordan coefficients for the group K. These Clebsch–Gordan coefficients can be taken for different orthogonal bases. Therefore, we obtain infinitesimal operators in different K bases. Correspondingly, we have the recurrence relations for Clebsch–Gordan coefficients of the decompositions (45) and (46) in different bases. The Clebsch–Gordan coefficients which are contained in the expressions for infinitesimal operators will be found for the most interesting bases in one of the forthcoming papers.

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